NUMERICAL APPLICATION OF ORDINARY DIFFERENTIAL EQUATIONS USING POWER SERIES FOR SOLVING HIGHER ORDER INITIAL VALUE PROBLEMS

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T. Y. Kyagya[1], D. Raymond[1] and J. Sabo[2]

[1]Department of Mathematics and Statistic, Federal University, Wukari-Nigeria

[2]Department of Mathematics, Adamawa University Mubi-Nigeria

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Abstract: In this research, we have proposed the numerical application of second derivative ordinary differential equations using power series for the direct solution of higher order initial value problems. The method was derived using power series, via interpolation and collocation procedure. The analysis of the method was studied, and it was found to be consistent, zero-stable and convergent. The derived method was able to solve highly stiff problems without converting to the equivalents system of first order ODEs. The generated results showed that the derived methods are notable better than those methods in literature. We further sketched the solution graph of our method and it is evident that the new method convergence toward the exact solution.

Keywords: Numerical application, ODEs, higher order IVPs, power series, collocation

Introduction

Mathematical modeling of real-life problems usually result into functional equation, for example, Ordinary differential equation and Partial differential equation, Integro and Integral differential equation, Stochastic differential equation and others. Not all ordinary differential equations such as those used to model real life problems can be solved analytically, Omar (2004).

Most of the problems in science, mathematical physics and engineering are formulated by differential equations. The solution of differential equations is a significant part to develop the various modeling in science and engineering. There are many analytical methods for finding the solution of ordinary differential equations. But a few numbers of differential equations have analytic solutions where a large numbers of differential equations have no analytic solutions.

In recent years, mathematical modeling of processes in biology, physics and medicine, particular in dynamic problems, cooling of a body and simple harmonic motion has led to significant scientific advances, both in mathematics and biosciences (Brauer & Castillo, 2012; Elazzouzi *et al.*, 2019). The applications of mathematics in biology and physics are completely opening new pathways of interactions, and this is certainly true, particular in areas like dynamic problems and cooling of a body.

This research considered the solution of high order initial value problems (IVPs) of ODEsof the form

$$y''=f(t,y,y'),\ y(a)=y_0,\ y'(a)=\eta_1 \qquad (1.1)$$

Equation (1.1) occurs in deferent fields of applied mathematics, among which are elasticity, fluid mechanics, and quantum mechanics as well as in engineering and physics. The existence and uniqueness of the solution for these equation have been discussed in Wend (1969). In general, finding the exact solutions of these equation is not easy. Over the years, deferent numerical methods have been developed in order to approximate the solution of equation (1.1). Among these methods are block method, linear multistep method, hybrid method and rung-kutta method, etc. (Lambert, 1973; Gear, 1966, 1971, 1978; Suleiman, 1979, 1989). Recently, some efforts have been made to develop hybrid block method for solving (1.1) directly; among others are Kuboye & Omar (2015), Omar & Abdelrahim (2016), Abdelrahim & Omar (2016), Alkasassbeh & Omar (2017), Skwame *et al.* (2019a, 2019b). However, these methods are focused on specific points (specifically, second order IVPs).

Mathematical Formulation of the Method

Power series polynomial of the form

$$y(t)=\sum_{j=0}^{p+q-1} a_j t^j \qquad (2.1)$$

is considered as a basis function to approximate the solution of the initial value problems of general second order ordinary differential equation of the form

$$y''=f(t,y,y'),\ y(a)=y_0,\ y'(a)=y_1 \qquad (2.2)$$

method is derived by the introduction of off-mesh points through one-step scheme following the method of Gragg and Stetter (1964), Gear (1964), Butcher (1965), and recently Omar & Adeyeye (2016), Omole & Ogunware (2018), Kamo *et al.* (2018), Skwame *et al.* (2019b).

Using (2.1) with $p=2$ and $q=7$, the polynomial is as follows;

$$y(t)=\sum_{j=0}^{8} a_j t^j \qquad (2.3)$$

Differentiating (2.3) twice, to yield

$$y''(t)=\sum_{j=0}^{8} j(j-1)a_j t^{j-2} \qquad (2.4)$$

Substituting (2.3) into (2.1) to yield

$$\sum_{j=0}^{8} j(j-1)a_j t^{j-2} = f(t,y,y') \qquad (2.5)$$

Now, interpolating (2.3) at $\frac{4}{9}$ *and* $\frac{5}{9}$ and collocating (2.5) at $0, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}$ *and* 1 lead to a system of equation written in a matrix form below;

$$TA=U \qquad (2.6)$$

$$\begin{bmatrix} t^0_{n+\frac{4}{9}} & t^1_{n+\frac{4}{9}} & t^2_{n+\frac{4}{9}} & t^3_{n+\frac{4}{9}} & t^4_{n+\frac{4}{9}} & t^5_{n+\frac{4}{9}} & t^6_{n+\frac{4}{9}} & t^7_{n+\frac{4}{9}} & t^8_{n+\frac{4}{9}} \\ t^0_{n+\frac{5}{9}} & t^1_{n+\frac{5}{9}} & t^2_{n+\frac{5}{9}} & t^3_{n+\frac{5}{9}} & t^4_{n+\frac{5}{9}} & t^5_{n+\frac{5}{9}} & t^6_{n+\frac{5}{9}} & t^7_{n+\frac{5}{9}} & t^8_{n+\frac{5}{9}} \\ 0 & 0 & 2t^0_n & 6t^1_n & 12t^2_n & 20t^3_n & 30t^4_n & 42t^5_n & 56t^6_n \\ 0 & 0 & 2t^0_{n+\frac{4}{9}} & 6t^1_{n+\frac{4}{9}} & 12t^2_{n+\frac{4}{9}} & 20t^3_{n+\frac{4}{9}} & 30t^4_{n+\frac{4}{9}} & 42t^5_{n+\frac{4}{9}} & 56t^6_{n+\frac{4}{9}} \\ 0 & 0 & 2t^0_{n+\frac{5}{9}} & 6t^1_{n+\frac{5}{9}} & 12t^2_{n+\frac{5}{9}} & 20t^3_{n+\frac{5}{9}} & 30t^4_{n+\frac{5}{9}} & 42t^5_{n+\frac{5}{9}} & 56t^6_{n+\frac{5}{9}} \\ 0 & 0 & 2t^0_{n+\frac{2}{3}} & 6t^1_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} & 30t^4_{n+\frac{2}{3}} & 42t^5_{n+\frac{2}{3}} & 56t^6_{n+\frac{2}{3}} \\ 0 & 0 & 2t^0_{n+\frac{7}{9}} & 6t^1_{n+\frac{7}{9}} & 12t^2_{n+\frac{7}{9}} & 20t^3_{n+\frac{7}{9}} & 30t^4_{n+\frac{7}{9}} & 42t^5_{n+\frac{7}{9}} & 56t^6_{n+\frac{7}{9}} \\ 0 & 0 & 2t^0_{n+\frac{8}{9}} & 6t^1_{n+\frac{8}{9}} & 12t^2_{n+\frac{8}{9}} & 20t^3_{n+\frac{8}{9}} & 30t^4_{n+\frac{8}{9}} & 42t^5_{n+\frac{8}{9}} & 56t^6_{n+\frac{8}{9}} \\ 0 & 0 & 2t^0_{n+1} & 6t^1_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} & 30t^4_{n+1} & 42t^5_{n+1} & 56t^6_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{4}{9}} \\ y_{n+\frac{5}{9}} \\ f_n \\ f_{n+\frac{4}{9}} \\ f_{n+\frac{5}{9}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{7}{9}} \\ f_{n+\frac{8}{9}} \\ f_{n+1} \end{bmatrix} \tag{2.7}$$

Using Gaussian elimination method, (2.7) is solved for the $a_j's$. The values of the $a_j's$ obtained are then substituted into (2.1), after some manipulations, this gives a continuous hybrid linear multistep method of the form;

$$y(t) = \alpha_{\frac{4}{9}}(t) y_{n+\frac{4}{9}} + \alpha_{\frac{5}{9}}(t) y_{n+\frac{5}{9}} + h^2 \left[\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_{v_i}(t) f_{n+v_i} \right], v_i = 0, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9} \tag{2.8}$$

Evaluating (2.8) non interpolating points to obtain the continuous form as,

$$\left.\begin{aligned}
&41150592 y_n - 205752960 y_{n+\frac{4}{9}} + 164602368 y_{n+\frac{5}{9}} = h^2 \begin{pmatrix} 312889 f_n + 26577810 f_{n+\frac{4}{9}} - 63742392 f_{n+\frac{5}{9}} + 79840824 f_{n+\frac{2}{3}} \\ -54941904 f_{n+\frac{7}{9}} + 20107773 f_{n+\frac{7}{9}} - 3074680 f_{n+1} \end{pmatrix} \\
&41150592 y_{n+\frac{2}{3}} + 411505920 y_{n+\frac{4}{9}} - 823011840 y_{n+\frac{5}{9}} = -h^2 \begin{pmatrix} 221 f_n - 408870 f_{n+\frac{4}{9}} - 4312728 f_{n+\frac{5}{9}} - 270312 f_{n+\frac{2}{3}} \\ -137232 f_{n+\frac{7}{9}} + 57393 f_{n+\frac{7}{9}} - 8792 f_{n+1} \end{pmatrix} \\
&137168640 y_{n+\frac{7}{9}} + 274337280 y_{n+\frac{4}{9}} - 411505920 y_{n+\frac{5}{9}} = -h^2 \begin{pmatrix} 137 f_n - 264222 f_{n+\frac{4}{9}} - 3049704 f_{n+\frac{5}{9}} - 1540392 f_{n+\frac{2}{3}} \\ -268272 f_{n+\frac{7}{9}} + 48573 f_{n+\frac{7}{9}} - 6440 f_{n+1} \end{pmatrix} \\
&68584320 y_{n+\frac{8}{9}} + 205752960 y_{n+\frac{4}{9}} - 274337280 y_{n+\frac{5}{9}} = -h^2 \begin{pmatrix} 95 f_n - 195426 f_{n+\frac{4}{9}} - 2329992 f_{n+\frac{5}{9}} - 1575672 f_{n+\frac{2}{3}} \\ -933552 f_{n+\frac{7}{9}} - 44037 f_{n+\frac{7}{9}} - 1736 f_{n+1} \end{pmatrix} \\
&41150592 y_{n+1} + 164602368 y_{n+\frac{4}{9}} - 205752960 y_{n+\frac{5}{9}} = -h^2 \begin{pmatrix} 95 f_n - 160146 f_{n+\frac{4}{9}} - 1857240 f_{n+\frac{5}{9}} - 1476888 f_{n+\frac{2}{3}} \\ -1032336 f_{n+\frac{7}{9}} - 516789 f_{n+\frac{7}{9}} - 37016 f_{n+1} \end{pmatrix}
\end{aligned}\right\} \tag{2.9}$$

Differentiating (2.8) once, yields

$$y'(t) = a'_{\frac{4}{9}}(t) y_{n+\frac{4}{9}} + \alpha'_{\frac{5}{9}}(t) y_{n+\frac{5}{9}} + h^2 \left[\sum_{j=0}^{1} \beta'_j(t) f_{n+j} + \beta'_{v_i}(t) f_{n+v_i} \right], v_i = 0, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9} \tag{2.10}$$

On evaluating (2.10) at all point, yield the discrete scheme as;

$$
\left.\begin{array}{l}
10160640hy'_n+91445760y_{n+\frac{4}{9}}-91445760y_{n+\frac{5}{9}}=-h^2\begin{pmatrix}955509f_n+36681610f_{n+\frac{4}{9}}-98376152f_{n+\frac{5}{9}}+126893144f_{n+\frac{2}{3}}\\-88948624f_{n+\frac{7}{9}}+32959913f_{n+\frac{8}{9}}-5085080f_{n+1}\end{pmatrix}\\
91445760hy'_{n\frac{4}{9}}+823011840y_{n+\frac{4}{9}}-823011840y_{n+\frac{5}{9}}=h^2\begin{pmatrix}955f_n-2607066f_{n+\frac{4}{9}}-3848040f_{n+\frac{5}{9}}+2232552f_{n+\frac{2}{3}}\\-1180656f_{n+\frac{7}{9}}+374031f_{n+\frac{8}{9}}-52136f_{n+1}\end{pmatrix}\\
91445760hy'_{n\frac{4}{9}}-823011840y_{n+\frac{4}{9}}-823011840y_{n+\frac{5}{9}}=-h^2\begin{pmatrix}731f_n-962010f_{n+\frac{4}{9}}-5350968f_{n+\frac{5}{9}}+1948296f_{n+\frac{2}{3}}\\-977616f_{n+\frac{7}{9}}+302967f_{n+\frac{8}{9}}-41720f_{n+1}\end{pmatrix}\\
1451520hy'_{n\frac{4}{9}}+13063680y_{n+\frac{4}{9}}-13063680y_{n+\frac{5}{9}}=-h^2\begin{pmatrix}3f_n-11162f_{n+\frac{4}{9}}-160616f_{n+\frac{5}{9}}-76312f_{n+\frac{2}{3}}+7184f_{n+\frac{7}{9}}\\-1105f_{n+\frac{8}{9}}+88f_{n+1}\end{pmatrix}\\
91445760hy'_{n\frac{4}{9}}+823011840y_{n+\frac{4}{9}}-823011840y_{n+\frac{5}{9}}=-h^2\begin{pmatrix}571f_n-828954f_{n+\frac{4}{9}}-9270072f_{n+\frac{5}{9}}-10787448f_{n+\frac{2}{3}}\\-4931280f_{n+\frac{7}{9}}+466263f_{n+\frac{8}{9}}-50680f_{n+1}\end{pmatrix}\\
91445760hy'_{n\frac{4}{9}}+823011840y_{n+\frac{4}{9}}-823011840y_{n+\frac{5}{9}}=-h^2\begin{pmatrix}29f_n-683046f_{n+\frac{4}{9}}-10086552f_{n+\frac{5}{9}}-8511720f_{n+\frac{2}{3}}\\-1253752f_{n+\frac{7}{9}}-3857679f_{n+\frac{8}{9}}+108480f_{n+1}\end{pmatrix}\\
10160640hy'_{n+1}+91445760y_{n+\frac{4}{9}}-91445760y_{n+\frac{5}{9}}=-h^2\begin{pmatrix}195f_n-121226f_{n+\frac{4}{9}}-888440f_{n+\frac{5}{9}}-1484728f_{n+\frac{2}{3}}\\-628816f_{n+\frac{7}{9}}-1607809f_{n+\frac{8}{9}}-349496f_{n+1}\end{pmatrix}
\end{array}\right\} \quad (2.11)
$$

Equation (2.9) and (2.11) can be written explicitly as

$$
\left.\begin{array}{l}
y_{n+\frac{4}{9}}=y_n+\frac{4}{9}hy'_n+\frac{2}{1607445}h^2\begin{pmatrix}27481f_n+770490f_{n+\frac{4}{9}}-2213568f_{n+\frac{5}{9}}+2901696f_{n+\frac{2}{3}}\\-2054016f_{n+\frac{7}{9}}+766017f_{n+\frac{8}{9}}-118720f_{n+1}\end{pmatrix}\\
y_{n+\frac{5}{9}}=y_n+\frac{5}{9}hy'_n+\frac{25}{164602368}h^2\begin{pmatrix}293921f_n+8952930f_{n+\frac{4}{9}}-25216632f_{n+\frac{5}{9}}+32907000f_{n+\frac{2}{3}}\\-23230800f_{n+\frac{7}{9}}+8648325f_{n+\frac{8}{9}}-1338680f_{n+1}\end{pmatrix}\\
y_{n+\frac{2}{3}}=y_n+\frac{2}{3}hy'_n+\frac{1}{79380}h^2\begin{pmatrix}4373f_n+139860f_{n+\frac{4}{9}}-388584f_{n+\frac{5}{9}}+506940f_{n+\frac{2}{3}}\\-357264f_{n+\frac{7}{9}}+132867f_{n+\frac{8}{9}}-20552f_{n+1}\end{pmatrix}\\
y_{n+\frac{7}{9}}=y_n+\frac{7}{9}hy'_n+\frac{49}{16796160}h^2\begin{pmatrix}22465f_n+741762f_{n+\frac{4}{9}}-2042712f_{n+\frac{5}{9}}+2668344f_{n+\frac{2}{3}}\\-1875600f_{n+\frac{7}{9}}+697221f_{n+\frac{8}{9}}-107800f_{n+1}\end{pmatrix}\\
y_{n+\frac{8}{9}}=y_n+\frac{8}{9}hy'_n+\frac{32}{1607445}h^2\begin{pmatrix}3817f_n+128898f_{n+\frac{4}{9}}-352800f_{n+\frac{5}{9}}+461328f_{n+\frac{2}{3}}\\-323136f_{n+\frac{7}{9}}+120330f_{n+\frac{8}{9}}-18592f_{n+1}\end{pmatrix}\\
y_{n+1}=y_n+hy'_n+\frac{1}{376320}h^2\begin{pmatrix}32527f_n+1116990f_{n+\frac{4}{9}}-3043656f_{n+\frac{5}{9}}+3983112f_{n+\frac{2}{3}}\\-2782512f_{n+\frac{7}{9}}+1041579f_{n+\frac{8}{9}}-159880f_{n+1}\end{pmatrix}\\
y'_{n+\frac{4}{9}}=y'_n+\frac{2}{178605}h\begin{pmatrix}8399f_n+319851f_{n+\frac{4}{9}}-868392f_{n+\frac{5}{9}}+1117452f_{n+\frac{2}{3}}\\-782928f_{n+\frac{7}{9}}+290052f_{n+\frac{8}{9}}-44744f_{n+1}\end{pmatrix}\\
y'_{n+\frac{5}{9}}=y'_n+\frac{5}{9144576}h\begin{pmatrix}171977f_n+6621930f_{n+\frac{4}{9}}-17600688f_{n+\frac{5}{9}}+22801800f_{n+\frac{2}{3}}\\-15991200f_{n+\frac{7}{9}}+5926725f_{n+\frac{8}{9}}-914480f_{n+1}\end{pmatrix}\\
y'_{n+\frac{2}{3}}=y'_n+\frac{1}{105840}h\begin{pmatrix}9953f_n+382914f_{n+\frac{4}{9}}-1013040f_{n+\frac{5}{9}}+1327368f_{n+\frac{2}{3}}\\-927072f_{n+\frac{7}{9}}+343413f_{n+\frac{8}{9}}-52976f_{n+1}\end{pmatrix}\\
y'_{n+\frac{7}{9}}=y'_n+\frac{7}{933120}h\begin{pmatrix}12535f_n+482454f_{n+\frac{4}{9}}-1277136f_{n+\frac{5}{9}}+1680504f_{n+\frac{2}{3}}\\-1159776f_{n+\frac{7}{9}}+431739f_{n+\frac{8}{9}}-66640f_{n+1}\end{pmatrix}\\
y'_{n+\frac{8}{9}}=y'_n+\frac{4}{178605}h\begin{pmatrix}4199f_n+161532f_{n+\frac{4}{9}}-427392f_{n+\frac{5}{9}}+561792f_{n+\frac{2}{3}}\\-384768f_{n+\frac{7}{9}}+146727f_{n+\frac{8}{9}}-22400f_{n+1}\end{pmatrix}\\
y'_{n+1}=y'_n+\frac{1}{62720}h\begin{pmatrix}5897f_n+227178f_{n+\frac{4}{9}}-601776f_{n+\frac{5}{9}}+792456f_{n+\frac{2}{3}}\\-545184f_{n+\frac{7}{9}}+213381f_{n+\frac{8}{9}}-29232f_{n+1}\end{pmatrix}
\end{array}\right\} \quad (2.12)
$$

Basic properties of the method

In this section, the analysis of the block method, which includes the order, error constant, consistency, zero stability, convergence and region of absolute stability region of the method shall be study.

Order and error constant

Let the linear operator defined on the method be $\ell[y(x);h]$, where,

$$\Delta\{y(x):h\}=A^{(0)}Y_m^{(i)}-\sum_{i=0}^{k}\frac{jh^{(i)}}{i!}y_n^{(i)}-h^{(3-1)}[d_i f(y_n)+b_iF(Y_m)], \tag{3.1}$$

Expanding Y_m *and* $F(Y_m)$ in Taylor series and comparing the coefficients of h gives

$$\Delta\{y(x):h\}=C_0y(x)+C_1y'(x)+\cdots+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\cdots \tag{3.2}$$

Definition 3.1: The linear operator L and the associate block method are said to be of order p if $C_0=C_1=\cdots=C_p=C_{p+1}=0,\quad C_{p+2}\neq 0.$ C_{p+2} is called the error constant. It implies that the local truncation error is given by $T_{n+k}=C_{p+2}h^{p+2}y^{p+3}(x)+0(h^{p+3})$

$$L\{y(x):h\}=C_0y(x)+C_1y'(x)+\cdots+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\cdots$$

Expanding the block in Taylor series expansion, gives

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{\left(\frac{4}{9}\right)^j}{j!}-y_n-\frac{4}{9}hy'_n-\frac{54962}{1607445}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{4892}{5103}\left(\frac{4}{9}\right)-\frac{7808}{2835}\left(\frac{5}{9}\right)+\frac{276352}{76545}\left(\frac{2}{3}\right)\\-\frac{456448}{178605}\left(\frac{7}{9}\right)+\frac{386}{405}\left(\frac{8}{9}\right)-\frac{6784}{45927}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{5}{9}\right)^j}{j!}-y_n-\frac{5}{9}hy'_n-\frac{7348025}{164602368}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{197375}{145152}\left(\frac{4}{9}\right)-\frac{1250825}{326592}\left(\frac{5}{9}\right)+\frac{4896875}{979776}\left(\frac{2}{3}\right)\\-\frac{149375}{42336}\left(\frac{7}{9}\right)+\frac{3431875}{2612736}\left(\frac{8}{9}\right)-\frac{85375}{419904}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j}{j!}-y_n-\frac{2}{3}hy'_n-\frac{4373}{79380}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{37}{21}\left(\frac{4}{9}\right)-\frac{514}{105}\left(\frac{5}{9}\right)+\frac{1207}{189}\left(\frac{2}{3}\right)\\-\frac{3308}{735}\left(\frac{7}{9}\right)+\frac{703}{420}\left(\frac{8}{9}\right)-\frac{734}{2835}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{7}{9}\right)^j}{j!}-y_n-\frac{7}{9}hy'_n-\frac{220157}{3359232}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{2019241}{933120}\left(\frac{4}{9}\right)-\frac{463393}{77760}\left(\frac{5}{9}\right)+\frac{5447869}{699840}\left(\frac{2}{3}\right)\\-\frac{127645}{23328}\left(\frac{7}{9}\right)+\frac{1265327}{622080}\left(\frac{8}{9}\right)-\frac{132055}{419904}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{\left(\frac{8}{9}\right)^j}{j!}-y_n-\frac{8}{9}hy'_n-\frac{122144}{1607445}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{21824}{8505}\left(\frac{4}{9}\right)-\frac{5120}{729}\left(\frac{5}{9}\right)+\frac{702976}{76545}\left(\frac{2}{3}\right)\\-\frac{382976}{59535}\left(\frac{7}{9}\right)+\frac{12224}{5103}\left(\frac{8}{9}\right)-\frac{84992}{229635}(1)\end{bmatrix}\\
\sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-hy'_n-\frac{32527}{376320}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{5319}{1793}\left(\frac{4}{9}\right)-\frac{18117}{2240}\left(\frac{5}{9}\right)+\frac{3387}{320}\left(\frac{2}{3}\right)\\-\frac{57969}{7840}\left(\frac{7}{9}\right)+\frac{49599}{17920}\left(\frac{8}{9}\right)-\frac{571}{1344}(1)\end{bmatrix}
\end{bmatrix} \tag{3.3}$$

Comparing the coefficient of h, the order p of the method and the error constant are given respectively by

$p=[5\quad 5\quad 5\quad 5\quad 5\quad 5]^T$ and

$$C_{p+2}=[1.7180\times10^{-7}\quad 1.7159\times10^{-7}\quad 1.7167\times10^{-7}\quad 1.7160\times10^{-7}\quad 1.7171\times10^{-7}\quad 1.7131\times10^{-7}]^T$$

Consistency of the method

A numerical method is said to be consistent if its order $p\geq 1$. Our method is thus consistent since it is of uniform order 5. Note that consistency controls the magnitude of the local truncation error committed at each stage of the computation, Fatunla (1988).

Zero stability of the method

Definition 3.2: the numerical method is said to be zero-stable, if the roots $z_s,\ s=1,2,\cdots,k$ of the first characteristics polynomial $\rho(z)$ defined by $\rho(z)=\det\left(zA^{(0)}-E\right)$ satisfies $|z_s|\le 1$ and every root satisfies $|z_s|=1$ have multiplicity not exceeding the order of the differential equation, Sunday (2018). The first characteristic polynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}z&0&0&0&0&-1\\0&z&0&0&0&-1\\0&0&z&0&0&-1\\0&0&0&z&0&-1\\0&0&0&0&z&-1\\0&0&0&0&0&z-1\end{bmatrix}\right|=z^5(z-1)$$

Thus, solving for z in

$$z^5(z-1) \tag{3.4}$$

gives $z=0,\ 0,0,0,\ 0,1$. Hence the block method is said to be zero stable.

Convergence of the block method

Theorem 3.1: the necessary and sufficient conditions for linear multistep method to be convergent are that it must be consistent and zero-stable. Hence our method formulated is consistent (Skwame *et al.*, 2019).

Region of absolute stability of our method

Definition 3.3: the region of absolute stability is the region of the complex z plane, where $z=\lambda h$ for which the method is absolute stable. To determine the region of absolute stability of the block method, the methods that compare neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. Thus, the method according to Yan (2011) is called the boundary locus method. Applying this methodon (2.12), we obtain the stability polynomial as

$$\begin{aligned}\bar{h}(w)=&\,h^{12}\left(\frac{653}{308629440}w^5+\frac{5}{177147}w^6\right)+h^{10}\left(-\frac{2509}{6200145}w^6-\frac{2653571}{129624364800}w^5\right)\\&+h^8\left(-\frac{27386717}{172832486400}w^5+\frac{12287}{2755620}w^6\right)+h^6\left(-\frac{13}{378}w^6-\frac{1459}{604800}w^5\right)+h^4\left(-\frac{138696553}{2133734400}w^5+\frac{625}{3402}w^6\right)\\&+h^2\left(-\frac{13}{21}w^6-\frac{19141}{75264}w^5\right)-2w^5+w^6\end{aligned}\tag{3.5}$$

On applying the stability polynomial (3.5), we obtain the region of absolute stability in Fig 1.

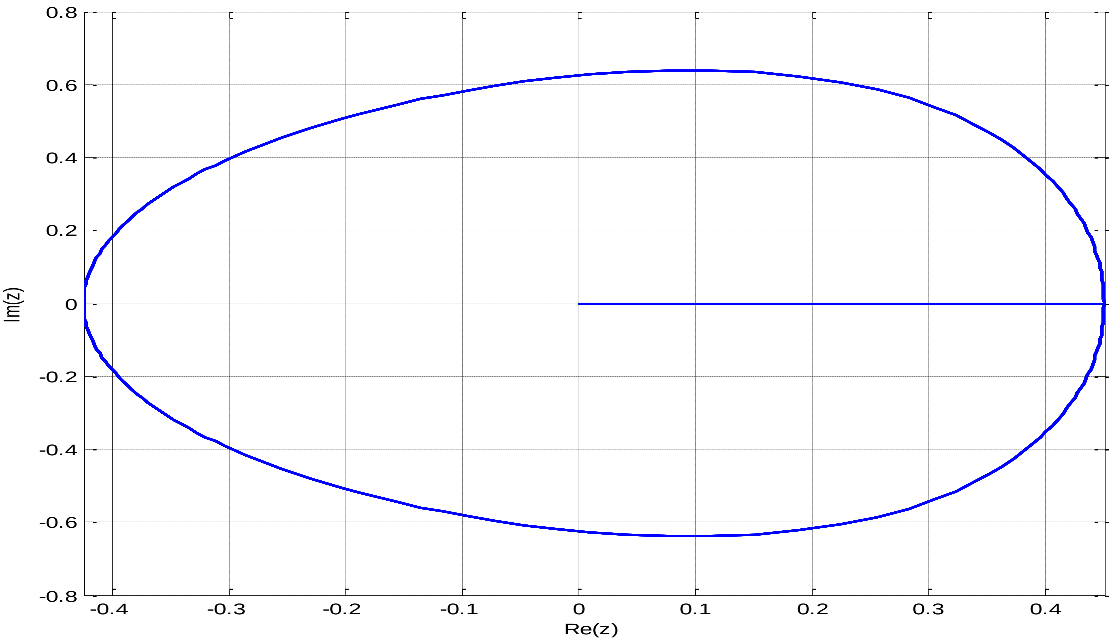

Fig. 1: Region of absolute stability of the method

Numerical implementation of the method

In this section, it is important to state that the method formulated can be used to implement differential equations of the form (1.1) without reducing it to an equivalent system of first order. We test the effectiveness and validity of our newly derived method by applying it to some real life problems and second order highly stiff initial value problems of the form (1.1). Our result are compared with the existing methods of Skwame *et al.* (2017), Areo & Omojola (2015), Omole & Ogunware (2018), Olanegan *et al.* (2018), Adeyefa *et al.* (2018), Kayode & Adegboro (2018).

Problem 1: Real-life problem

Dynamic problems

A $10kg$ mass is attached to a spring having a spring constant of $140\,N/M$. The mass is set in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(t)=5\sin t$. Find the subsequent motion of the mass $(t:0.10\le t\le 1.00)$ if the force due to air resistance is $90\left(\frac{dx}{dt}\right)N$.

Where $m=10, k=140, a=90\; and\; F(t)=5\sin t$

The equation can be written as

$$dsolver\left(\left\{\frac{d^2x}{dt^2}+9\frac{dx}{dt}+14x(t)=\frac{1}{2}\sin(t),\ x(0)=0,\ x'(0)=-1\right\}\right)$$

With analytic solution is given by

$$x(t)=\frac{1}{500}\left(-90e^{-2t}+99e^{-7t}+13\sin t-9\cos t\right)$$

Source: Skwame *et al.* (2017) and Areo & Omojola (2015).

Table 1: Showing the comparison of Problem 1

X	Exact Result	Computed Result	Error in our Method	Error in Skwame *et al.* (2017)	Error in Areo & Omojola (2015)
0.1	-0.06436205154552458248	-0.06436205102813963251	5.1739e-10	1.0647e-07	1.2744e-08
0.2	-0.08430720522644774945	-0.08430720423489605205	9.9155e-10	1.1870e-06	3.0442e-08
0.3	-0.08405225313390041905	-0.08405225388560049594	7.5170e-10	2.2635e-06	4.1501e-08
0.4	-0.07529304213333374810	-0.07529304281786442748	6.8453e-10	2.8219e-06	4.5385e-08
0.5	-0.06357063960355798563	-0.06357063934273678993	2.6082e-10	2.9539e-06	4.4298e-08
0.6	-0.05142117069384508163	-0.05142117055138588829	1.4246e-10	2.8187e-06	4.0466e-08
0.7	-0.03993052956438697070	-0.03993052856491394937	9.9947e-10	2.5466e-06	3.5475e-08
0.8	-0.02949865862803573900	-0.02949865777324012431	8.5480e-10	2.2235 -06	3.0285e-08
0.9	-0.02021269131259124546	-0.02021269059230590628	7.2030e-10	1.8991e-06	2.5408e-08
1.0	-0.01202699425403169607	-0.01202699365301409334	6.0102e-10	1.5988e-06	2.1071e-08

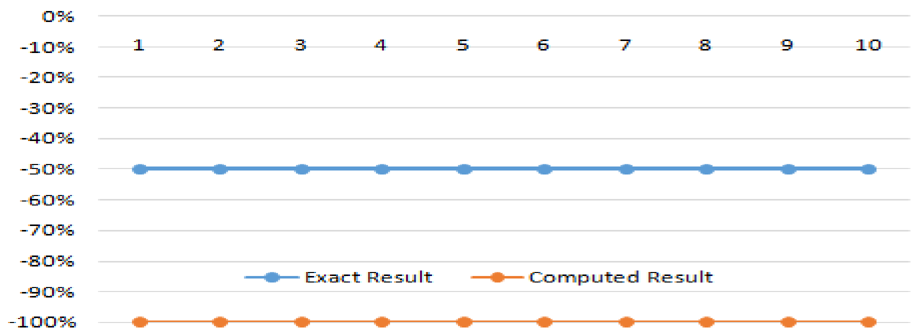

Fig. 2: Graphical solution of Problem 1

Problem 2: ***Real-life problem***

Cooling of a body

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2}+\frac{dy}{dt}=0$. By using the substitution $z=\frac{dy}{dt}$ or otherwise, find y in terms of t given that $y=60$ when $t=0$ and $y=35$ when $t=6\ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulation of the Problem.

$$y''=\frac{-y'}{3},\ y(0)=60,\ y'(0)=-\frac{80}{9},\ h=0.1$$

With analytic solution is given by

$$y(x)=\frac{80}{3}e^{-\left(\frac{1}{3}\right)x}+\frac{100}{3}.$$

Source: Omole & Ogunware (2018) and Olanegan *et al.* (2018).

Table 2: Showing the comparison of problem 2

X	Exact Result	Computed Result	Error in our Method	Error in Omole & Ogunware (2018)	Error in Olanegan *et al.* (2018)
0.1	59.125762679520157387 00	59.125762679520157387 00	0.0000	3.5500e-11	7.4764e-06
0.2	58.280186267509806339 00	58.280186267509806339 00	0.0000	4.5800e-11	2.9394e-05
0.3	57.462331147625588617 00	57.462331147625588617 00	0.0000	7.0000e-11	6.4802e-05
0.4	56.671288507811932106 00	56.671288507811932106 00	0.0000	6.5000e-12	1.1279e-05
0.5	55.906179330416375307 00	55.906179330416375307 00	0.0000	3.3300e-11	1.7250e-04
0.6	55.166153415412849564 00	55.166153415412849564 00	0.0000	4.2000e-11	2.4310e-04
0.7	54.450388435647511050 00	54.450388435647511050 00	0.0000	4.3800e-11	3.2383e-04
0.8	53.758089023057298472 00	53.758089023057298472 00	0.0000	1.0700e-10	4.1393e-04
0.9	53.088485884845809762 00	53.088485884845809762 00	0.0000	6.5800e-11	5.1271e-04
1.0	52.440834948634380011 00	52.440834948634380011 00	0.0000	1.6900e-10	6.1951e-04

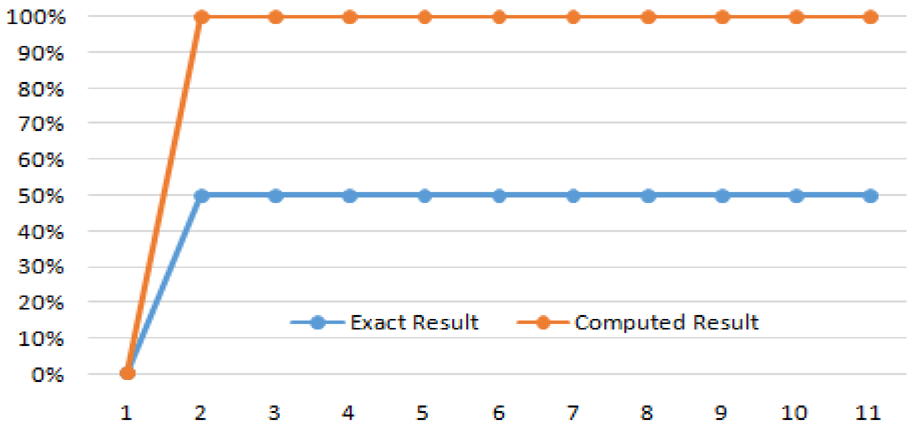

Fig. 3: Graphical solution of Problem 2

Problem 3

Consider a highly stiff linear second order problem

$$y'' = y', \quad y(0) = 0, \; y'(0) = -1, \; h = 0.1$$

With analytic solution is given by

$$y(x) = 1 - \exp(x).$$

Source: Omole & Ogunware (2018), Adeyefa *et al.* (2018), Kayode & Adegboro (2018).

Table 3: Showing the comparison of Problem 3

X	Exact Result	Computed Result	Error in our Method	Error in Omole & Ogunware (2018)	Error in Adeyefa *et al.* (2018)	Error in Kayode & Adegboro (2018)
0.1	-0.1051709180756476248	-0.1051709180756474609 8	1.6382e-16	7.5650e-11	3.2482e-12	-
0.2	-0.2214027581601698339	-0.2214027581601692801 2	5.5378e-16	1.6017e-10	8.5643e-11	3.4602e-09
0.3	-0.3498588075760031040	-0.3498588075760018883 2	1.2157e-15	1.7600e-10	3.4401e-10	5.6760e-09
0.4	-0.4918246976412703178	-0.4918246976412681155 2	2.2023e-15	6.0784e-10	7.4251e-10	7.6413e-09
0.5	-0.6487212707001281468	-0.6487212707001245721 3	3.5747e-15	1.4729e-09	1.3785e-09	1.0497e-08
0.6	-0.8221188003905089749	-0.8221188003905035717 5	5.4032e-15	2.5336e-09	2.2193e-09	1.4495e-08
0.7	-1.0137527074704765216	-1.0137527074704687534 0	7.7682e-15	4.7876e-09	3.3875e-09	1.8782e-08
0.8	-1.2255409284924676046	-1.2255409284924568418 0	1.0763e-14	7.2770e-09	4.8470e-09	2.2799e-08
0.9	-1.4596031111569496638	-1.4596031111569351709 0	1.4493e-14	1.0170e-08	6.7518e-09	2.8258e-08
1.0	-1.7182818284590452354	-1.7182818284590261550 0	1.9080e-14	1.4827e-08	9.0628e-09	3.5547e-08

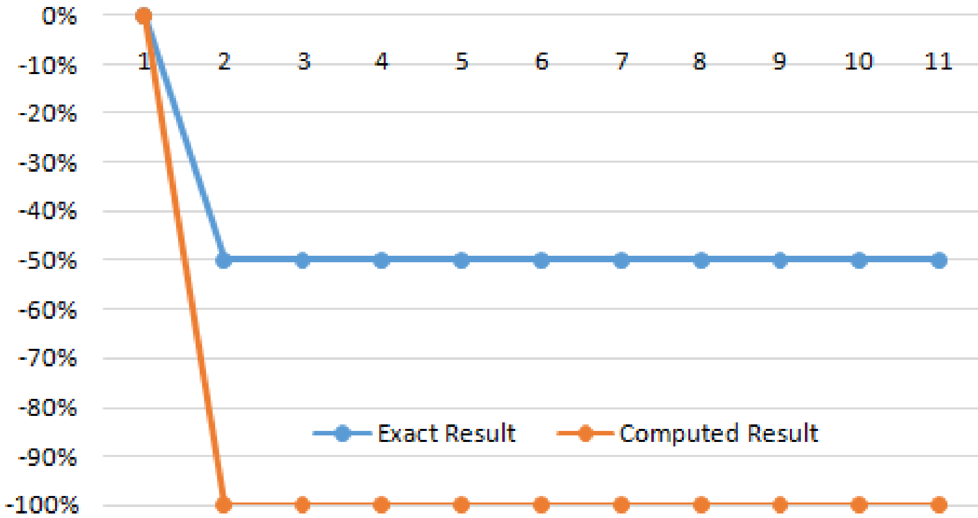

Fig. 3: Graphical solution of Problem 3

Discussion and Conclusion

In this research, we have proposed the numerical application of second derivative ordinary differential equations for the direct solution of higher order initial value problems. The method was derived using power series, via interpolation and collocation procedure. The analysis of the method was studied, and it was found to be consistent, zero-stable and convergent. The derived method able to solve some highly stiff second ODEs problems without converting to the equivalents system of first order ODEs. The generated results, as appear in the tables 4.1-4.3, shown that the derived methods are more accurate than the existing method of Skwame *et al.* (2017), Areo & Omojola (2015), Omole & Ogunware (2018), Olanegan *et al.* (2018), Adeyefa *et al.* (2018), Kayode & Adegboro (2018).We further sketched the solution graph of our method and it is evident that the new method convergence toward the exact solution.

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